Decoupling the Multiconductor Transmission Line Equations

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***Abstract*— A comprehensive discussion of the method of decoupling the multiconductor transmission line (MTL) equations by the method of transformation of the voltages and currents to mode voltages and currents in order to obtain their general solution is presented. Various ways of defining and obtaining the transformations are shown which serve to connect the myriad of such definitions and also point out where inconsistencies in those definitions can result. Structures for which the decoupling is assured are also discussed. The MTL equations to be decoupled are in the frequency domain, and extensions to their applicability in the time-domain are shown.**

I. Introduction

IN THIS PAPER we consider a $(n+1)$-conductor line consisting of $(n+1)$ conductors which are parallel to the z axis in a rectangular coordinate system. Let us assume that the conductors are of uniform cross section in the z direction as are the properties of the surrounding media (which may be inhomogeneous). In other words, the line cross-sectional dimensions are independent of z. Such a line is said to be uniform. The multiconductor transmission line (MTL) equations for frequency-domain analysis (sinusoidal, steady-state excitation of the line) are

$$\frac{d}{dz}\hat{\boldsymbol{V}}(z) = -\hat{\boldsymbol{Z}}\hat{\boldsymbol{I}}(z) \tag{1a}$$

$$\frac{d}{dz}\hat{\boldsymbol{I}}(z) = -\hat{\boldsymbol{Y}}\hat{\boldsymbol{V}}(z) \tag{1b}$$

where $\hat{\boldsymbol{V}}(z)$ and $\hat{\boldsymbol{I}}(z)$ are $n \times 1$ vectors containing the phasor line voltages (with respect to the zero-th or reference conductor) and phasor line currents, respectively. We use a caret ($\hat{}$) to denote complex-valued quantities. The $n \times n$ complex matrices of per-unit-length impedance, $\hat{\boldsymbol{Z}}$, and admittance, $\hat{\boldsymbol{Y}}$, are symmetric and contain the $n \times n$ real, symmetric matrices of per-unit-length resistance, $\boldsymbol{R}$, inductance, $\boldsymbol{L}$, conductance, $\boldsymbol{G}$, and capacitance, $\boldsymbol{C}$, as

$$\hat{\boldsymbol{Z}} = \boldsymbol{R} + j\omega\boldsymbol{L} \tag{2a}$$

$$\hat{\boldsymbol{Y}} = \boldsymbol{G} + j\omega\boldsymbol{C}. \tag{2b}$$

The matrices $\boldsymbol{L}, \boldsymbol{C}$, and $\boldsymbol{G}$ are also positive definite as may be shown from energy considerations. The fundamental assumption in modeling a MTL with the transmission line equations is that the electric and magnetic fields lie in a plane transverse to the z axis which is called the transverse electromagnetic field structure or mode (TEM) of propagation. This field structure is identical to that of the static field in the transverse plane. This allows the computation of the per-unit-length matrices $\boldsymbol{G}, \boldsymbol{L}$, and $\boldsymbol{C}$ as solutions of Laplace's equation for the static 2D field structure in the transverse plane. If the surrounding medium is homogeneous with parameters of conductivity, σ, permittivity, ε, and permeability, μ, then $\boldsymbol{G}, \boldsymbol{L}$, and $\boldsymbol{C}$ satisfy the following identities:

$$\boldsymbol{LC} = \boldsymbol{CL} = \mu\varepsilon\boldsymbol{1}_n \tag{3a}$$

$$\boldsymbol{LG} = \boldsymbol{GL} = \mu\sigma\boldsymbol{1}_n \tag{3b}$$

where $\boldsymbol{1}_n$ is the $n \times n$ identity matrix with one's on the main diagonal and zero's elsewhere. If the medium surrounding the conductors is inhomogeneous, these identities obviously do not apply. All of these per-unit-length parameter matrices can be functions of frequency although the per-unit-length resistance, $\boldsymbol{R}$, typically has the strongest dependence on frequency, and the frequency dependence of $\boldsymbol{L}$ and $\boldsymbol{C}$ is typically negligible. In the case of imperfect conductors ($\boldsymbol{R} \neq \boldsymbol{0}$) the $n \times n$ internal inductance matrix, $\boldsymbol{L}_i$, contains the internal inductances of the conductors and is added to $\boldsymbol{L}$. At high frequencies, $\boldsymbol{R}$ increases as $\sqrt{f}$ whereas $\boldsymbol{L}_i$ decreases at a rate of $\sqrt{f}$. Hence the internal inductance matrix is frequently smaller that $\boldsymbol{L}$ and therefore often neglectable. For the case of imperfect conductors and/or inhomogeneous surrounding media, the TEM mode cannot exist. In this case it is assumed that these fields remain approximately TEM which is referred to as the quasi-TEM approximation. The validity of the quasi-TEM approximation was investigated in [1], [2], and the MTL equations have been successfully used to characterize lossy and/or inhomogeneous structures into the gigahertz frequency range.

These coupled transmission line equations have a long history of representing many diverse structures. Numerous texts have documented their utility [3]–[14]. Some of the more pioneering work was done around 1940 by Pipes [15], [16]. Pipes also gave a thorough discussion of their solution for uniform and nonuniform lines in [17]. Subsequent applications of the MTL equations appeared in telephone system [18], [19] and power distribution system analyzes [20]–[25]. The emphasis on prediction of crosstalk in cables that interconnect electronic equipment renewed that interest [26]–[30], and the MTL equations were also adapted to the investigation of the effects of incident fields on those cable systems [30]–[32]. The increasing emphasis on microwave circuits provided a renewed interest in using the MTL equations to model these high-frequency structures that continues today. Much of that work concentrated on lossless lines [33]–[37]. The increasing

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frequencies of use in high-density circuits have caused the conductor losses, represented by $\boldsymbol{R}, \boldsymbol{L}_i$, and, to some extent, losses in the medium represented by $\boldsymbol{G}$ to be significant. The frequency-domain transfer function obtained from a solution of the phasor MTL equations in (1) is a straightforward way of including those frequency-dependent losses and can be used to provide the time-domain solution for general MTL's via the inverse Fourier transform. The input signal to the line is decomposed into its spectral components and passed through the phasor transfer function yielding the Fourier transform of the output signal of the line. This is converted to the time domain with the inverse Fourier transform. This time-domain solution technique has a long history of use and is referred to as the time-domain to frequency-domain or TDFD method [13]. The only drawback to the TDFD method is that it relies on superposition and hence cannot be used in the case of nonlinear terminations of the line since the transfer function must contain those terminations and hence is nonlinear. The increasing use of nonlinear line terminations has required the direct solution of the complete MTL equations in the time domain and has resulted in numerous techniques such as finite-difference time-domain (FDTD) methods [13], [38], the waveform relaxation technique [39], the generalized method of characteristics [40], [41] and the asymptotic waveform evaluation (AWE) technique [42], [43]. Another reason for the development of alternative direct time-domain solution techniques is that the decoupling method which we will discuss requires similarity transformations which, for the lossy line case, are functions of frequency thereby making their direct application to the time-domain solution of the MTL equations difficult. However, the frequency-domain results of this paper can be applied to the time domain for lossy lines with nonlinear terminations by generating a linear 2n-port of the line and using convolution [44]. So the frequency-domain decoupling of the MTL equations has broad application. The decoupling method has also been applied to nonuniform lines [45], [46]. Although we will discuss the exact solution of the phasor MTL equations, there are also various approximate ways of solving them most of which make lumped-circuit approximations to the MTL [13], [14], [47], and [48].

The purpose of this paper is to give a comprehensive discussion of the primary method of solving the phasor MTL equations via the method of decoupling. Although this has been a standard technique for over 60 years there remain some misunderstandings and inconsistencies in its application. Some of these were highlighted in [49]–[51]. Here we give an alternative view of these problems and discuss structures to which the decoupling technique applies.

II. Decoupling the MTL Equations

The method of using a change of variables is perhaps the most frequently-used technique for generating the general solution to the MTL equations. In implementing that method we transform to mode quantities as

$$\hat{\boldsymbol{V}}(z) = \hat{\boldsymbol{T}}_V \hat{\boldsymbol{V}}_m(z) \tag{4a}$$

$$\hat{\boldsymbol{I}}(z) = \hat{\boldsymbol{T}}_I \hat{\boldsymbol{I}}_m(z). \tag{4b}$$

The $n \times n$ complex matrices $\hat{\boldsymbol{T}}_V$ and $\hat{\boldsymbol{T}}_I$ define a change of variables between the actual phasor line voltages and currents, $\hat{\boldsymbol{V}}$ and $\hat{\boldsymbol{I}}$, and the mode voltages and currents, $\hat{\boldsymbol{V}}_m$ and $\hat{\boldsymbol{I}}_m$. In order for this to be valid, these $n \times n$ matrices must be nonsingular, i.e., $\hat{\boldsymbol{T}}_V^{-1}$ and $\hat{\boldsymbol{T}}_I^{-1}$ must exist where we denote the inverse of an $n \times n$ matrix $\boldsymbol{M}$ as $\boldsymbol{M}^{-1}$, in order to go between both sets of variables. Substituting these into the phasor MTL equations in (1) gives

$$\frac{d}{dz}\hat{\boldsymbol{V}}_m = -\hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Z}}\hat{\boldsymbol{T}}_I\hat{\boldsymbol{I}}_m \tag{5a}$$

$$\frac{d}{dz}\hat{\boldsymbol{I}}_m = -\hat{\boldsymbol{T}}_I^{-1}\hat{\boldsymbol{Y}}\hat{\boldsymbol{T}}_V\hat{\boldsymbol{V}}_m. \tag{5b}$$

If we can obtain a $\hat{\boldsymbol{T}}_V$ and a $\hat{\boldsymbol{T}}_I$ such that $\hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Z}}\hat{\boldsymbol{T}}_I$ and $\hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Y}}\hat{\boldsymbol{T}}_V$ are diagonal as

$$\hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Z}}\hat{\boldsymbol{T}}_I = \hat{\boldsymbol{z}} = \begin{bmatrix} \hat{z}_1 & 0 & \cdots & 0 \\ 0 & \hat{z}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{z}_n \end{bmatrix} \tag{6a}$$

$$\hat{\boldsymbol{T}}_I^{-1}\hat{\boldsymbol{Y}}\hat{\boldsymbol{T}}_V = \hat{\boldsymbol{y}} = \begin{bmatrix} \hat{y}_1 & 0 & \cdots & 0 \\ 0 & \hat{y}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{y}_n \end{bmatrix} \tag{6b}$$

then the phasor MTL equations are uncoupled as

$$\begin{gathered} \frac{d}{dz}\hat{V}_{m1}(z) = -\hat{z}_1\hat{I}_{m1}(z), \quad \frac{d}{dz}\hat{I}_{m1}(z) = -\hat{y}_1\hat{V}_{m1}(z) \\ \vdots \\ \frac{d}{dz}\hat{V}_{mn}(z) = -\hat{z}_n\hat{I}_{mn}(z), \quad \frac{d}{dz}\hat{I}_{mn}(z) = -\hat{y}_n\hat{V}_{mn}(z). \end{gathered} \tag{7}$$

Therefore if we can find two $n \times n$ matrices $\hat{\boldsymbol{T}}_V$ and $\hat{\boldsymbol{T}}_I$ which simultaneously diagonalize both per-unit-length parameter matrices, $\hat{\boldsymbol{Z}}$ and $\hat{\boldsymbol{Y}}$, the solution essentially reduces to the solution of n uncoupled two-conductor lines.

In order to further address that question, we examine the application of the mode transformations to the uncoupled, second-order MTL equations obtained from (1) by differentiating each with respect to z and substituting the other:

$$\frac{d^2}{dz^2}\hat{\boldsymbol{V}}(z) = \hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}\hat{\boldsymbol{V}}(z) \tag{8a}$$

$$\frac{d^2}{dz^2}\hat{\boldsymbol{I}}(z) = \hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}\hat{\boldsymbol{I}}(z). \tag{8b}$$

In differentiating each equation with respect to z we are assuming that the per-unit-length parameter matrices, $\hat{\boldsymbol{Z}}$ and $\hat{\boldsymbol{Y}}$, are independent of z, i.e., the line is uniform. It is important to note that $\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}} \neq \hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}$ and the order of multiplication must be preserved. Substituting the transformations given in (4) yields

$$\begin{aligned} \frac{d^2}{dz^2}\hat{\boldsymbol{V}}_m(z) &= \hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}\hat{\boldsymbol{T}}_V\hat{\boldsymbol{V}}_m(z) \\ &= (\hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Z}}\hat{\boldsymbol{T}}_I)(\hat{\boldsymbol{T}}_I^{-1}\hat{\boldsymbol{Y}}\hat{\boldsymbol{T}}_V)\hat{\boldsymbol{V}}_m(z) \end{aligned}$$

$$= \hat{z}\hat{y}\hat{V}_m(z) \tag{9a}$$

$$\begin{aligned}\frac{d^2}{dz^2}\hat{I}_m(z) &= \hat{T}_I^{-1}\hat{Y}\hat{Z}\hat{T}_I\hat{I}_m(z)\\ &= (\hat{T}_I^{-1}\hat{Y}\hat{T}_V)(\hat{T}_V^{-1}\hat{Z}\hat{T}_I)\hat{I}_m(z)\\ &= \hat{y}\hat{z}\hat{I}_m(z).\end{aligned} \tag{9b}$$

If $\hat{z}$ and $\hat{y}$ are each diagonalized by $\hat{T}_V$ and $\hat{T}_I$ as in (6), then the second-order equations in (9) are likewise diagonalized by $\hat{T}_V$ and $\hat{T}_I$ but the reverse is not necessarily true. For example, if $\hat{Z}$ and $\hat{Y}$ happen to satisfy the condition that $\hat{Z}\hat{Y} = \hat{Y}\hat{Z} = \hat{\alpha}\mathbf{1}_n$ (an important special case of a lossless line in a homogeneous medium) then we may choose $\hat{T}_V = \hat{T}_I = \mathbf{1}_n$ and the second-order differential equations in (9) are uncoupled yet the first-order equations in (5) are not: $\hat{z} = \hat{Z}$ and $\hat{y} = \hat{Y}$. For this important special case, all n propagation constants are identical. In the following we will show that if all n propagation constants are distinct, then transformations $\hat{T}_V$ and $\hat{T}_I$ can be found which decouple the second-order equations in (9), and, in addition, $\hat{z}$ and $\hat{y}$ are likewise diagonal so that the first-order equations in (5) are *simultaneously* decoupled by the same transformations $\hat{T}_V$ and $\hat{T}_I$. In the case of repeated propagation constants we will show that $\hat{z}$ and $\hat{y}$ are not necessarily diagonal but are block diagonal where the blocks are associated with the distinct propagation constants.

The decoupling of the second-order equations as in (9) relies on finding a $\hat{T}_V$ and a $\hat{T}_I$ which diagonalize $\hat{Z}\hat{Y}$ and $\hat{Y}\hat{Z}$ via similarity transformations as

$$\hat{T}_V^{-1}\hat{Z}\hat{Y}\hat{T}_V = \hat{z}\hat{y} = \hat{\gamma}^2 \tag{10a}$$

$$\hat{T}_I^{-1}\hat{Y}\hat{Z}\hat{T}_I = \hat{y}\hat{z} = \hat{\gamma}^2 \tag{10b}$$

where $\hat{\gamma}^2$ is a $n \times n$ diagonal matrix. The columns of $\hat{T}_V$ are said to be the eigenvectors of $\hat{Z}\hat{Y}$ and the columns of $\hat{T}_I$ are the eigenvectors of $\hat{Y}\hat{Z}$ [52–54]. The entries in $\hat{\gamma}^2, \hat{\gamma}_i^2$ for $i = 1, \cdots, n$, are the eigenvalues of $\hat{Z}\hat{Y}$ and of $\hat{Y}\hat{Z}$ [52–54]. That the eigenvalues of $\hat{Z}\hat{Y}$ and $\hat{Y}\hat{Z}$ are the same follows from the fact that the eigenvalues of a matrix, $\boldsymbol{M}$, and its transpose, $\boldsymbol{M}^t$, are the same [52–54]. Taking the transpose of $\hat{Z}\hat{Y}$ yields $(\hat{Z}\hat{Y})^t = \hat{Y}^t\hat{Z}^t = \hat{Y}\hat{Z}$ where we have used the fact that $\hat{Z}$ and $\hat{Y}$ are symmetric, i.e., $\hat{Z}^t = \hat{Z}$ and $\hat{Y}^t = \hat{Y}$. Therefore the transpose of $\hat{Z}\hat{Y}$ is $\hat{Y}\hat{Z}$ thus showing that they have the same eigenvalues. Hence, in order to decouple the second-order MTL equations we only need to find a $\hat{T}_V$ or a $\hat{T}_I$ that diagonalize the product of $\hat{Z}$ and $\hat{Y}$ as in (10a) or as in (10b).

In order to diagonalize $\hat{Z}\hat{Y}$ or $\hat{Y}\hat{Z}$ as in (10) we must be able to find a linearly independent set of n eigenvectors (columns of $\hat{T}_V$ and $\hat{T}_I$) in which case $\hat{T}_V$ and $\hat{T}_I$ are nonsingular [52,53]. A sufficient condition for this is that all n eigenvalues, $\hat{\gamma}_i^2$, are distinct [52,53]. The case of n distinct eigenvalues is straightforward and poses no problems since it can be shown that the eigenvectors associated with distinct eigenvalues are unique only within an arbitrary constant [52,53]. The problems arise in the case of repeated eigenvalues of $\hat{Z}\hat{Y}$. There are important cases where some of the eigenvalues are repeated yet a linearly independent set of n eigenvectors can be found such that $\hat{Z}\hat{Y}$ can be diagonalized as in (10). In the case of repeated eigenvalues, the eigenvectors (columns of $\hat{T}_V$ and $\hat{T}_I$) corresponding to those repeated eigenvalues are not so unique. The eigenvectors associated with a repeated set of eigenvalues can be transformed with a nonsingular transformation to another set which retain the ability to decouple the second-order equations. Structures that exhibit certain types of symmetry can result in repeated eigenvalues and hence give rise to this nonunique assignment of the columns of $\hat{T}_V$ or $\hat{T}_I$ associated with those repeated eigenvalues. The nonunique assignment of these columns of $\hat{T}_V$ or $\hat{T}_I$ will not effect the diagonalization of the second-order equations in (10) but will affect the diagonalization of the first-order equations in (6). Nevertheless, the problem of decoupling the first-order equations in (1) is closely associated with the problem of decoupling the second-order equations of (8). Hence we will concentrate on decoupling the second-order equations in (8) and will assume in this article that $\hat{Z}\hat{Y}$ can be diagonalized as in (10).

Thus the equations governing the mode voltages and currents in (9) are decoupled and have the simple solution

$$\hat{V}_m(z) = e^{-\hat{\gamma}z}\hat{V}_m^+ + e^{\hat{\gamma}z}\hat{V}_m^- \tag{11a}$$

$$\hat{I}_m(z) = e^{-\hat{\gamma}z}\hat{I}_m^+ - e^{\hat{\gamma}z}\hat{I}_m^- \tag{11b}$$

where the matrix exponentials are defined as

$$e^{\pm\hat{\gamma}z} = \begin{bmatrix} e^{\pm\hat{\gamma}_1 z} & 0 & \cdots & 0 \\ 0 & e^{\pm\hat{\gamma}_2 z} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\pm\hat{\gamma}_n z} \end{bmatrix} \tag{12}$$

and $\hat{V}_m^\pm$ and $\hat{I}_m^\pm$ are $n\times 1$ vectors of (as yet) undetermined constants associated with the forward/backward-traveling waves of the modes. Transforming back to the actual line voltages and currents via (4) gives the general solution to the MTL equations as

$$\hat{V}(z) = \hat{T}_V(e^{-\hat{\gamma}z}\hat{V}_m^+ + e^{\hat{\gamma}z}\hat{V}_m^-) \tag{13a}$$

$$\hat{I}(z) = \hat{T}_I(e^{-\hat{\gamma}z}\hat{I}_m^+ - e^{\hat{\gamma}z}\hat{I}_m^-). \tag{13b}$$

Therefore if we can find a transformation that diagonalizes either $\hat{Z}\hat{Y}$ or $\hat{Y}\hat{Z}$ then the decoupling of the second-order equations is assured and the general solution to the MTL equations in (1) can be readily obtained.

Because of (10), the mode transformations $\hat{T}_V$ and $\hat{T}_I$ are related. To obtain this relationship, suppose there are k distinct eigenvalues. Arrange them in $\hat{\gamma}^2$ as

$$\hat{\gamma}^2 = \begin{bmatrix} \hat{\gamma}_1^2\mathbf{1}_{n1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\gamma}_2^2\mathbf{1}_{n2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \hat{\gamma}_k^2\mathbf{1}_{nk} \end{bmatrix} \tag{14}$$

where $\mathbf{1}_{nk}$ is the $nk \times nk$ identity matrix and nk is the multiplicity of the repeated eigenvalue $\hat{\gamma}_k^2$. Digital computer subroutines that compute eigenvalues/eigenvectors of a general matrix do not generally provide this ordering. The eigenvectors (columns of $\hat{T}_V$ and $\hat{T}_I$) are arranged in the same sequence as the eigenvalues in (14) as

$$\hat{T}_V = [\hat{T}_{V1} \quad \hat{T}_{V2} \quad \cdots \quad \hat{T}_{Vk}] \tag{15a}$$
$$\hat{T}_I = [\hat{T}_{I1} \quad \hat{T}_{I2} \quad \cdots \quad \hat{T}_{Ik}] \tag{15b}$$

where $\hat{T}_{Vk}$ and $\hat{T}_{Ik}$ are $n \times nk$. Using the relations in (10) yields

$$\hat{Z}\hat{Y} = \hat{T}_V \hat{\gamma}^2 \hat{T}_V^{-1} \tag{16a}$$
$$\hat{Y}\hat{Z} = \hat{T}_I \hat{\gamma}^2 \hat{T}_I^{-1}. \tag{16b}$$

Observing that $\hat{Z}$ and $\hat{Y}$ as well as $\hat{\gamma}^2$ are symmetric and taking the transpose of (16) yields

$$\hat{\gamma}^2 \hat{T}_V^t \hat{T}_I = \hat{T}_V^t \hat{T}_I \hat{\gamma}^2 \tag{17a}$$
$$\hat{\gamma}^2 \hat{T}_I^t \hat{T}_V = \hat{T}_I^t \hat{T}_V \hat{\gamma}^2. \tag{17b}$$

Because of these relations and the assumption that $\hat{\gamma}_i^2 \neq \hat{\gamma}_j^2$ for $i \neq j$, the $\hat{T}_V$ and $\hat{T}_I$ must be related as

$$\hat{T}_V^t \hat{T}_I = \hat{D} = \begin{bmatrix} \hat{D}_1 & 0 & \cdots & 0 \\ 0 & \hat{D}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{D}_k \end{bmatrix} \tag{18}$$

where $\hat{D}$ is block diagonal and $\hat{D}_k$ is $nk \times nk$. Since $\hat{T}_V$ and $\hat{T}_I$ are nonsingular, $\hat{D}$ is also nonsingular. However, $\hat{D}$ is not necessarily symmetric. Equation (18) shows that

$$\hat{T}_{Ii}^t \hat{T}_{Vj} = \hat{T}_{Vi}^t \hat{T}_{Ij} = 0 \qquad i \neq j. \tag{19}$$

If $\hat{T}_V$ and $\hat{T}_I$ are real, (19) is equivalent to stating that the eigenvectors are orthogonal [52]–[54]. If all n eigenvalues are distinct, each $\hat{D}_k$ is a scalar and $\hat{D}$ is diagonal (and therefore symmetric) and hence $\hat{T}_I^t \hat{T}_V = \hat{D} = \hat{T}_V^t \hat{T}_I$ with $\hat{D}_i$ on the main diagonal and zeros elsewhere.

The transformation matrices can be redefined such that $\hat{D}$ is the identity matrix. An essential requirement of any such redefinition is that the redefined transformations must retain the ability to diagonalize the second-order equations as in (10). There are several ways of doing this. For example, suppose we redefine the transformations as

$$\hat{T}_V' = \hat{T}_V (\hat{D}^t)^{-1} \tag{20a}$$
$$\hat{T}_I' = \hat{T}_I. \tag{20b}$$

The new transformations yield

$$(\hat{T}_V')^t \hat{T}_I' = \hat{D}^{-1} \hat{T}_V^t \hat{T}_I = \mathbf{1}_n. \tag{21}$$

These redefined transformations retain the ability to decouple the second-order equations. To show this, we form $(\hat{T}_V')^{-1} \hat{Z}\hat{Y} \hat{T}_V' = \hat{D}^t \hat{T}_V^{-1} \hat{Z}\hat{Y} \hat{T}_V (\hat{D}^t)^{-1} = \hat{D}^t \hat{\gamma}^2 (\hat{D}^t)^{-1} = \hat{\gamma}^2$. This is true because $\hat{D}$ and $\hat{\gamma}^2$ commute since they are block diagonal as in (14) and (18) and each block of $\hat{\gamma}^2$ is simply $\hat{\gamma}_i^2 \mathbf{1}_{ni}$. Because the redefined transformations retain the ability to decouple the second-order equations, all of the previous results remain unchanged. Hence, we may assume throughout the remainder of this article that the transformations are chosen such that $\hat{D}$ is the identity matrix:

$$\hat{T}_V^t \hat{T}_I = \hat{T}_I^t \hat{T}_V = \mathbf{1}_n. \tag{22}$$

Next we will show that $\hat{z}$ and $\hat{y}$ in (6) are block diagonal and symmetric. Let us assume that none of the eigenvalues (propagation constants) are zero and hence, according to (10), $\hat{z}$ and $\hat{y}$ are nonsingular. Premultiplying and postmultiplying (10) by, for example, $\hat{y}^{-1}$ and $\hat{z}^{-1}$ yields

$$\hat{z}\hat{\gamma}^2 = \hat{\gamma}^2 \hat{z} \tag{23a}$$
$$\hat{y}\hat{\gamma}^2 = \hat{\gamma}^2 \hat{y}. \tag{23b}$$

Observing that $\hat{\gamma}^2$ has the form given in (14) where $\hat{\gamma}_i^2 \neq \hat{\gamma}_j^2$ for $i \neq j$, (23a) shows that $\hat{z}$ is block diagonal as

$$z = \begin{bmatrix} \hat{z}_{11} & 0 & \cdots & 0 \\ 0 & \hat{z}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{z}_{nknk} \end{bmatrix} \tag{24}$$

where $\hat{z}_{ij}$ is $ni \times nj$. Similarly we can show that $\hat{y}$ is also block diagonal. In the case of n distinct eigenvalues, the $\hat{z}_{ii}$ are scalars and hence $\hat{z}$ and $\hat{y}$ are diagonal matrices so that the first-order equations in (6) are in fact decoupled. In the case of repeated eigenvalues $\hat{z}$ and $\hat{y}$ are simply block diagonal matrices. Nevertheless, if the transformations are chosen such that (22) is satisfied then $\hat{z}$ and $\hat{y}$ are also symmetric. This can be shown by substituting (22) into (6) to yield

$$\hat{T}_I^t \hat{Z} \hat{T}_I = \hat{z} \tag{25a}$$
$$\hat{T}_V^t \hat{Y} \hat{T}_V = \hat{y}. \tag{25b}$$

Since $\hat{Z}$ and $\hat{Y}$ are symmetric, this shows that $\hat{z}$ and $\hat{y}$ are also symmetric, i.e., $\hat{z} = \hat{z}^t$ and $\hat{y} = \hat{y}^t$.

The general solutions for the line voltages and currents given in (13) contain a total of $4n$ undetermined constants in the $n \times 1$ vectors $\hat{V}_m^+, \hat{V}_m^-, \hat{I}_m^+$, and $\hat{I}_m^-$. We will now relate those by defining the characteristic impedance matrix thereby reducing the number of undetermined constants to $2n$. Substituting (13b) into (1b) yields

$$\begin{aligned} \hat{V}(z) &= -\hat{Y}^{-1} \frac{d}{dz} \hat{I}(z) \\ &= \underbrace{\hat{Y}^{-1} \hat{T}_I \hat{\gamma} \hat{T}_I^{-1}}_{\hat{Z}_C} \hat{T}_I (e^{-\hat{\gamma} z} \hat{I}_m^+ + e^{\hat{\gamma} z} \hat{I}_m^-). \end{aligned} \tag{26}$$

If we define the characteristic impedance matrix as

$$\hat{Z}_C = \hat{Y}^{-1} \hat{T}_I \hat{\gamma} \hat{T}_I^{-1} \tag{27}$$

then

$$\hat{V}(z) = \hat{Z}_C \hat{T}_I (e^{-\hat{\gamma} z} \hat{I}_m^+ + e^{\hat{\gamma} z} \hat{I}_m^-) \tag{28a}$$
$$\hat{I}(z) = \hat{T}_I (e^{-\hat{\gamma} z} \hat{I}_m^+ - e^{\hat{\gamma} z} \hat{I}_m^-) \tag{28b}$$

and the number of unknowns is reduced to the $2n$ unknowns in the $n \times 1$ vectors $\hat{I}_m^+$ and $\hat{I}_m^-$. An alternative form of the characteristic impedance matrix can be obtained from (10b)

$$\hat{Y}^{-1} \hat{T}_I = \hat{Z} \hat{T}_I \hat{\gamma}^{-2} \tag{29}$$

Substituting this into (27) yields

$$\hat{Z}_C = \hat{Z}\hat{T}_I\hat{\gamma}^{-1}\hat{T}_I^{-1}. \tag{30}$$

Similarly, substituting (13a) into (1a) yields

$$\begin{aligned}\hat{I}(z) &= -\hat{Z}^{-1}\frac{d}{dz}\hat{V}(z)\\ &= \underbrace{\hat{Z}^{-1}\hat{T}_V\hat{\gamma}\hat{T}_V^{-1}}_{\hat{Y}_C}\hat{T}_V(e^{-\hat{\gamma}z}\hat{V}_m^+ - e^{\hat{\gamma}z}\hat{V}_m^-).\end{aligned} \tag{31}$$

If we define the characteristic admittance matrix as

$$\hat{Y}_C = \hat{Z}^{-1}\hat{T}_V\hat{\gamma}\hat{T}_V^{-1} \tag{32}$$

then

$$\hat{V}(z) = \hat{T}_V(e^{-\hat{\gamma}z}\hat{V}_m^+ + e^{\hat{\gamma}z}\hat{V}_m^-) \tag{33a}$$

$$\hat{I}(z) = \hat{Y}_C\hat{T}_V(e^{-\hat{\gamma}z}\hat{V}_m^+ - e^{\hat{\gamma}z}\hat{V}_m^-) \tag{33b}$$

and the number of unknowns is reduced to the $2n$ unknowns in the $n \times 1$ vectors $\hat{V}_m^+$ and $\hat{V}_m^-$. An alternative form of the characteristic admittance matrix can be obtained from (10a)

$$\hat{Z}^{-1}\hat{T}_V = \hat{Y}\hat{T}_V\hat{\gamma}^{-2}. \tag{34}$$

Substituting this into (32) yields

$$\hat{Y}_C = \hat{Y}\hat{T}_V\hat{\gamma}^{-1}\hat{T}_V^{-1}. \tag{35}$$

Additional relations for the characteristic impedance/admittance matrix can be obtained. Substituting (28a) and (33b) into (1a) gives

$$\hat{V}_m^{\pm} = e^{\pm\hat{\gamma}z}\hat{T}_V^{-1}\hat{Y}_C^{-1}\hat{Z}^{-1}\hat{Z}_C\hat{T}_I\hat{\gamma}e^{\mp\hat{\gamma}z}\hat{I}_m^{\pm}. \tag{36a}$$

Similarly, substituting (28b) and (33a) into (1b) gives

$$\hat{V}_m^{\pm} = e^{\pm\hat{\gamma}z}\hat{T}_V^{-1}\hat{Y}^{-1}\hat{T}_I\hat{\gamma}e^{\mp\hat{\gamma}z}\hat{I}_m^{\pm}. \tag{36b}$$

Equation (36) shows that

$$\hat{Z}_C = \hat{Z}\hat{Y}_C\hat{Y}^{-1}. \tag{37}$$

Substituting (32) into (37) yields

$$\hat{Z}_C = \hat{T}_V\hat{\gamma}\hat{T}_V^{-1}\hat{Y}^{-1}. \tag{38}$$

But this is the inverse of (35) showing that $\hat{Y}_C = \hat{Z}_C^{-1}$ as expected. In like manner we may obtain

$$\hat{Z}_C = \hat{T}_V\hat{\gamma}^{-1}\hat{T}_V^{-1}\hat{Z} \tag{39}$$

$$\hat{Y}_C = \hat{T}_I\hat{\gamma}^{-1}\hat{T}_I^{-1}\hat{Y} \tag{40}$$

$$\hat{Y}_C = \hat{T}_I\hat{\gamma}\hat{T}_I^{-1}\hat{Z}^{-1}. \tag{41}$$

Additional relations for the characteristic impedance/admittance matrices can be obtained by substituting (18). For example, substituting (18) into (27) yields

$$\hat{Z}_C = \hat{Y}^{-1}(\hat{T}_V^t)^{-1}\hat{\gamma}\hat{T}_V^t \tag{42}$$

since $\hat{D}\hat{\gamma}\hat{D}^{-1} = \hat{\gamma}$ even if $\hat{T}_V$ and $\hat{T}_I$ are not chosen such that (22) is satisfied. But (42) is the transpose of (38) showing that, as expected, the characteristic impedance and admittance matrices are symmetric: $\hat{Z}_C^t = \hat{Z}_C, \hat{Y}_C^t = \hat{Y}_C$. Similarly, substituting (18) into (30), (32) and (35) yields

$$\hat{Z}_C = \hat{Z}(\hat{T}_V^t)^{-1}\hat{\gamma}^{-1}\hat{T}_V^t \tag{43}$$

$$\hat{Y}_C = \hat{Z}^{-1}(\hat{T}_I^t)^{-1}\hat{\gamma}\hat{T}_I^t \tag{44}$$

$$\hat{Y}_C = \hat{Y}(\hat{T}_I^t)^{-1}\hat{\gamma}^{-1}\hat{T}_I^t. \tag{45}$$

TABLE I

Characteristic Impedance/Admittance Matrix Relations

$\hat{\mathbf{Z}}_C$	$\hat{\mathbf{Y}}_C$
$\hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}_I\hat{\gamma}\hat{\mathbf{T}}_I^{-1}$	$\hat{\mathbf{Z}}^{-1}\hat{\mathbf{T}}_V\hat{\gamma}\hat{\mathbf{T}}_V^{-1}$
$\hat{\mathbf{Z}}\hat{\mathbf{T}}_I\hat{\gamma}^{-1}\hat{\mathbf{T}}_I^{-1}$	$\hat{\mathbf{Y}}\hat{\mathbf{T}}_V\hat{\gamma}^{-1}\hat{\mathbf{T}}_V^{-1}$
$\hat{\mathbf{T}}_V\hat{\gamma}^{-1}\hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}$	$\hat{\mathbf{T}}_I\hat{\gamma}^{-1}\hat{\mathbf{T}}_I^{-1}\hat{\mathbf{Y}}$
$\hat{\mathbf{T}}_V\hat{\gamma}\hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Y}}^{-1}$	$\hat{\mathbf{T}}_I\hat{\gamma}\hat{\mathbf{T}}_I^{-1}\hat{\mathbf{Z}}^{-1}$
$\hat{\mathbf{Y}}^{-1}\left(\hat{\mathbf{T}}_V^t\right)^{-1}\hat{\gamma}\hat{\mathbf{T}}_V^t$	$\hat{\mathbf{Z}}^{-1}\left(\hat{\mathbf{T}}_I^t\right)^{-1}\hat{\gamma}\hat{\mathbf{T}}_I^t$
$\hat{\mathbf{Z}}\left(\hat{\mathbf{T}}_V^t\right)^{-1}\hat{\gamma}^{-1}\hat{\mathbf{T}}_V^t$	$\hat{\mathbf{Y}}\left(\hat{\mathbf{T}}_I^t\right)^{-1}\hat{\gamma}^{-1}\hat{\mathbf{T}}_I^t$

A summary of the relations for the characteristic impedance/admittance matrices is presented in Table I.

III. Specialized Relations

All of the above expressions for voltage and current in (28) and (33) as well as the expressions for the characteristic impedance/admittance matrix given in Table I involve $\hat{T}_V$ or $\hat{T}_I$. None involve $\hat{T}_V$ and $\hat{T}_I$. They only assume that the products of the per-unit-length impedance and admittance matrices, $\hat{Z}$ and $\hat{Y}$ are diagonalizable as in (10) and that $\hat{Z}$ and $\hat{Y}$ are symmetric. So long as one consistently uses expressions for $\hat{Z}_C$ as in Table I and the expressions for voltage and current in (28) or (33) that involve only $\hat{T}_V$ or $\hat{T}_I$ the results are independent of the redefinition of the elements of $\hat{T}_V$ or $\hat{T}_I$ such as in (20). In addition, they do not assume that the matrices $\hat{z}$ and $\hat{y}$ in (6) or $\hat{D}$ in (18) are diagonal or that $\hat{T}_V$ and $\hat{T}_I$ are chosen to satisfy (22). The problems and inconsistencies arise when one attempts to define relations between $\hat{T}_V$ and $\hat{T}_I$ so that the above relations can be written in terms of a mix of $\hat{T}_V$ and $\hat{T}_I$.

It is common to find the following relations [12], [34]–[37], [49]–[51]:

$$\hat{Z}_C = \hat{M}_V\hat{M}_V^t = (\hat{M}_I^t)^{-1}\hat{M}_I^{-1} = \hat{M}_V\hat{M}_I^{-1} \tag{46a}$$

$$\begin{aligned}\hat{M}_V^t\hat{M}_I &= \mathbf{1}_n\\ &= \hat{M}_I^t\hat{M}_V\end{aligned} \tag{46b}$$

$$\hat{M}_I = \hat{Z}^{-1}\hat{M}_V\hat{\gamma} \tag{46c}$$

$$\hat{M}_V = \hat{Y}^{-1}\hat{M}_I\hat{\gamma}. \tag{46d}$$

The $\hat{M}_V$ and $\hat{M}_I$ are not the same as $\hat{T}_V$ and $\hat{T}_I$ but are related by appropriate transformations of the columns of those matrices as will be demonstrated.

In order to obtain the relationships in (46) we will assume that $\hat{T}_V$ and $\hat{T}_I$ are chosen to satisfy (22), $\hat{T}_V^t\hat{T}_I = \mathbf{1}_n$, and will define a nonsingular, block diagonal, matrix $\hat{\boldsymbol{d}}$ such that $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{y}}$ are block normalized to the propagation constants of the modes, $\hat{\gamma}$, as

$$\hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Z}}\hat{\boldsymbol{T}}_I = \hat{\boldsymbol{z}} = \hat{\boldsymbol{d}}\hat{\gamma} \tag{47a}$$

$$\hat{\boldsymbol{T}}_I^{-1}\hat{\boldsymbol{Y}}\hat{\boldsymbol{T}}_V = \hat{\boldsymbol{y}} = \hat{\boldsymbol{d}}^{-1}\hat{\gamma} \tag{47b}$$

where $\hat{\boldsymbol{d}}_{ij}$ is $ni \times nj$ and $\hat{\boldsymbol{d}}_{ij} = \boldsymbol{0}$ for $i \neq j$. This is permissible since $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{y}}$ are similarly block diagonal as was shown previously and we assume that there are no zero propagation constants. In addition we showed that $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{y}}$ are symmetric and hence $\hat{\boldsymbol{d}}$ is symmetric, i.e., $\hat{\boldsymbol{d}} = \hat{\boldsymbol{d}}^t$. Therefore $\hat{\boldsymbol{T}}_V$ and $\hat{\boldsymbol{T}}_I$ are related as

$$\hat{\boldsymbol{T}}_I = \hat{\boldsymbol{Z}}^{-1}\hat{\boldsymbol{T}}_V\hat{\boldsymbol{d}}\hat{\gamma} \tag{48a}$$

$$\hat{\boldsymbol{T}}_V = \hat{\boldsymbol{Y}}^{-1}\hat{\boldsymbol{T}}_I\hat{\boldsymbol{d}}^{-1}\hat{\gamma}. \tag{48b}$$

Substituting these into the relations for the characteristic impedance matrix given in Table I and observing that the transformations are chosen to satisfy (22) yields

$$\hat{\boldsymbol{Z}}_C = \hat{\boldsymbol{T}}_V\hat{\boldsymbol{d}}\hat{\boldsymbol{T}}_I^{-1} = \hat{\boldsymbol{T}}_V\hat{\boldsymbol{d}}\hat{\boldsymbol{T}}_V^t. \tag{49}$$

Define the transformations in (46) as

$$\hat{\boldsymbol{M}}_V = \hat{\boldsymbol{T}}_V\hat{\boldsymbol{D}}_V \tag{50a}$$

$$\hat{\boldsymbol{M}}_I = \hat{\boldsymbol{T}}_I\hat{\boldsymbol{D}}_I \tag{50b}$$

where $\hat{\boldsymbol{D}}_V$ and $\hat{\boldsymbol{D}}_I$ are block diagonal, nonsingular matrices with $\hat{\boldsymbol{D}}_{V,Iij}$ of dimension $ni \times nj$ and $\hat{\boldsymbol{D}}_{V,Iij} = \boldsymbol{0}$ for $i \neq j$. In terms of these, the above results become

$$\hat{\boldsymbol{Z}}_C = \hat{\boldsymbol{M}}_V\hat{\boldsymbol{D}}_V^{-1}\hat{\boldsymbol{d}}\hat{\boldsymbol{D}}_I\hat{\boldsymbol{M}}_I^{-1} = \hat{\boldsymbol{M}}_V\hat{\boldsymbol{D}}_V^{-1}\hat{\boldsymbol{d}}(\hat{\boldsymbol{D}}_V^{-1})^t\hat{\boldsymbol{M}}_V^t \tag{51a}$$

$$\hat{\boldsymbol{M}}_V^t\hat{\boldsymbol{M}}_I = \hat{\boldsymbol{D}}_V^t\hat{\boldsymbol{D}}_I \tag{51b}$$

$$\hat{\boldsymbol{M}}_I^t\hat{\boldsymbol{M}}_V = \hat{\boldsymbol{D}}_I^t\hat{\boldsymbol{D}}_V \tag{51c}$$

$$\hat{\boldsymbol{M}}_I = \hat{\boldsymbol{Z}}^{-1}\hat{\boldsymbol{M}}_V\hat{\boldsymbol{D}}_V^{-1}\hat{\boldsymbol{d}}\hat{\gamma}\hat{\boldsymbol{D}}_I \tag{51d}$$

$$\hat{\boldsymbol{M}}_V = \hat{\boldsymbol{Y}}^{-1}\hat{\boldsymbol{M}}_I\hat{\boldsymbol{D}}_I^{-1}\hat{\boldsymbol{d}}^{-1}\hat{\gamma}\hat{\boldsymbol{D}}_V. \tag{51e}$$

In order to provide the equivalence between these expressions and the common ones given in (46) we must have

$$\hat{\boldsymbol{D}}_I = \hat{\boldsymbol{d}}^{-1/2} \tag{52a}$$

$$\hat{\boldsymbol{D}}_V = \hat{\boldsymbol{d}}^{1/2} \tag{52b}$$

in which case (51) are identical to (46). It should be noted that in the common case where all n eigenvalues are distinct, $\hat{\boldsymbol{d}}$ is diagonal and these transformation matrices are simple to obtain.

Perhaps the desire to obtain the relations in (46) arises for the following reason. The average power flow on the line is

$$P_{\mathrm{av}}(z) = \tfrac{1}{2}\mathrm{Re}(\hat{\boldsymbol{V}}^t\hat{\boldsymbol{I}}^*) = \tfrac{1}{2}\mathrm{Re}(\hat{\boldsymbol{V}}_m^t\hat{\boldsymbol{T}}_V^t\hat{\boldsymbol{T}}_I^*\hat{\boldsymbol{I}}_m^*) \tag{53}$$

where $\hat{\boldsymbol{M}}^*$ denotes the complex conjugate of $\hat{\boldsymbol{M}}$. If $\hat{\boldsymbol{T}}_V$ and $\hat{\boldsymbol{T}}_I$ are such that

$$\hat{\boldsymbol{T}}_V^t\hat{\boldsymbol{T}}_I^* = \boldsymbol{1}_n \tag{54}$$

then

$$P_{\mathrm{av}}(z) = \tfrac{1}{2}\mathrm{Re}(\hat{\boldsymbol{V}}_m^t\hat{\boldsymbol{I}}_m^*) \tag{55}$$

and the power is the sum of the powers of the individual modes. The condition in (54) is equivalent to stating that the voltage and current eigenvectors, the columns of $\hat{\boldsymbol{T}}_V$ and $\hat{\boldsymbol{T}}_I$, are orthonormal [52]–[54]. In subsequent subsections we will show several cases for which the MTL equations are decoupleable and (54) is satisfied. For these cases $\hat{\boldsymbol{M}}_V^t\hat{\boldsymbol{M}}_I^* \neq \boldsymbol{1}_n$ except in the lossless line case where $\hat{\boldsymbol{M}}_V$ and $\hat{\boldsymbol{M}}_I$ are both real so that (46b) is satisfied.

IV. Solution for Line Categories

The solution process described previously assumes that one can find n × n, nonsingular transformation matrices, $\hat{\boldsymbol{T}}_I$ and/or $\hat{\boldsymbol{T}}_V$, which diagonalize the product of per-unit-length parameter matrices, $\hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}$ or $\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}$, via a similarity transformation as $\hat{\boldsymbol{T}}_I^{-1}\hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}\hat{\boldsymbol{T}}_I = \hat{\gamma}^2$ and $\hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}\hat{\boldsymbol{T}}_V = \hat{\gamma}^2$. There are a number of known cases of $n \times n$ matrices, $\hat{\boldsymbol{M}}$, whose diagonalization is assured via a similarity transformation as $\hat{\boldsymbol{T}}^{-1}\hat{\boldsymbol{M}}\hat{\boldsymbol{T}}$.

These are [52]–[54]:

1) all eigenvalues of $\hat{\boldsymbol{M}}$ are distinct;
2) $\hat{\boldsymbol{M}}$is real, and symmetric;
3) $\hat{\boldsymbol{M}}$is complex but normal, i.e., $\hat{\boldsymbol{M}}\hat{\boldsymbol{M}}^{t^*} = \hat{\boldsymbol{M}}^{t^*}\hat{\boldsymbol{M}}$ where we denote the transpose of a matrix by t and its conjugate by *;
4) $\hat{\boldsymbol{M}}$is complex and Hermitian, i.e., $\hat{\boldsymbol{M}} = \hat{\boldsymbol{M}}^{t^*}$.

For normal or Hermitian $\hat{\boldsymbol{M}}$, the transformation matrix can be found such that $\hat{\boldsymbol{T}}^{-1} = \hat{\boldsymbol{T}}^{t^*}$ which is said to be a unitary transformation. For a real, symmetric $\boldsymbol{M}$, the transformation matrix can be found such that $\boldsymbol{T}^{-1} = \boldsymbol{T}^t$ which is said to be an orthogonal transformation. For other types of matrices, we are not assured that a nonsingular transformation can be found that diagonalizes it.

There exist digital computer subroutines that find the eigenvalues and eigenvectors of a general complex matrix. These can be used to attempt to diagonalize $\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}$ or $\hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}$. However, because the number of conductors, n, of the MTL can be quite large, it is important to investigate the conditions under which we can obtain an efficient and numerically stable diagonalization. In addition, the diagonalization must be repeated at each frequency so it is important to determine where $\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}$ or $\hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}$ can be diagonalized with a frequency-independent transformation. For application to the direct time-domain solution of the MTL equations via decoupling we also must require the transformation to be real and frequency independent. The following subsections address those points.

A. Perfect Conductors in Lossy, Homogeneous Media

For this case, $\boldsymbol{R} = \boldsymbol{0}$ and we have the identities in (3) so that

$$\hat{\boldsymbol{Z}} = j\omega\boldsymbol{L} \tag{56a}$$

$$\begin{aligned}\hat{Y} &= G + j\omega C \\ &= \mu(\sigma + j\omega\varepsilon)L^{-1}\end{aligned} \tag{56b}$$

and

$$\hat{Y}\hat{Z} = \hat{Z}\hat{Y} = j\omega\mu(\sigma + j\omega\varepsilon)1_n. \tag{57}$$

The propagation constants are identical and given by $\hat{\gamma}_i = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$. Hence we only need to diagonalize L as

$$T^{-1}LT = L_m \tag{58}$$

where L_m is a diagonal matrix with eigenvalues l_{mi} on the diagonal and zero's elsewhere. This can be readily accomplished with an orthogonal transformation such that $T^{-1} = T^t$ using, for example, the stable and highly efficient Jacobi algorithm [54]. Defining the transformation matrices as

$$\begin{aligned}\hat{T}_V &= \hat{T}_I \\ &= T \\ &= (T^{-1})^t\end{aligned} \tag{59}$$

we have

$$\begin{aligned}\hat{T}_V^{-1}\hat{Z}\hat{T}_I &= j\omega L_m \\ &= \hat{d}\hat{\gamma}\end{aligned} \tag{60a}$$

$$\begin{aligned}\hat{T}_I^{-1}\hat{Y}\hat{T}_V &= \mu(\sigma + j\omega\varepsilon)L_m^{-1} \\ &= \hat{d}^{-1}\hat{\gamma}.\end{aligned} \tag{60b}$$

Hence $\hat{z}$ and $\hat{y}$ are diagonal. Therefore we obtain

$$\hat{d} = \frac{j\omega}{\hat{\gamma}}L_m \tag{61a}$$

$$\hat{T}_V^t\hat{T}_I = 1_n. \tag{61b}$$

Thus

$$\hat{D}_V = \sqrt{\frac{j\omega}{\hat{\gamma}}}L_m^{1/2} \tag{62a}$$

$$\hat{D}_I = \hat{D}_V^{-1} \tag{62b}$$

so that the transformations in (50) become

$$\hat{M}_V = \sqrt{\frac{j\omega}{\hat{\gamma}}}TL_m^{1/2} \tag{63a}$$

$$\hat{M}_I = (\hat{M}_V^{-1})^t \tag{63b}$$

and the power flow relation in (55) holds. In addition, $\hat{M}_V^t\hat{M}_I^* = 1_n$ in the case of a lossless medium, $\sigma = 0$. One can also verify that the identities in (46) hold. The characteristic impedance becomes

$$\hat{Z}_C = \frac{j\omega}{\hat{\gamma}}L. \tag{64}$$

An example of this case is the coupled stripline.

B. Perfect Conductors in Lossless, Inhomogeneous Media

For this case, $R = 0$ and $G = 0$ and $\hat{Y}\hat{Z} = -\omega^2CL$. However, we no longer have the identity in (3). Nevertheless we may diagonalize L and C since they are real, symmetric and positive definite. This is a classic and well-known problem [52–54]. First find an orthogonal transformation that diagonalizes C as

$$U^tCU = \theta^2 \tag{65}$$

where θ^2 is a diagonal matrix with θ_i^2 on the main diagonal and zero's elsewhere and $U^{-1} = U^t$. Since C is real, symmetric, and positive definite, its eigenvalues, θ_i^2, are real, positive, and nonzero. Hence, we can obtain the square root of θ^2, θ, which is real, diagonal, and nonsingular and form the product $\theta U^tLU\theta$. Since this is real and symmetric, we can diagonalize it with another orthogonal transformation as

$$S^t(\theta U^tLU\theta)S = \Lambda^2 \tag{66}$$

where Λ^2 is a diagonal matrix with real elements Λ_i^2 on the main diagonal and zero's elsewhere and $S^{-1} = S^t$ as before. Define the matrix T as

$$T = U\theta S. \tag{67}$$

In order to minimize numerical errors, the columns of T can be normalized to a Euclidean length of unity as

$$T_{\text{norm}} = T\alpha \tag{68}$$

where α is the $n \times n$ diagonal matrix with entries

$$\alpha_{ii} = \frac{1}{\sqrt{\sum_{k=1}^{n}T_{ki}^2}} \tag{69a}$$

$$\alpha_{ij} = 0. \tag{69b}$$

The mode transformations that simultaneously diagonalize L and C can then be defined as

$$\begin{aligned}\hat{T}_I &= U\theta S\alpha \\ &= T_{\text{norm}}\end{aligned} \tag{70a}$$

$$\begin{aligned}\hat{T}_V &= U\theta^{-1}S\alpha^{-1} \\ &= T_{\text{norm}}^{t^{-1}}.\end{aligned} \tag{70b}$$

Also

$$\begin{aligned}\hat{T}_I^{-1} &= \alpha^{-1}S^t\theta^{-1}U^t \\ &= \hat{T}_V^t\end{aligned} \tag{71a}$$

$$\begin{aligned}\hat{T}_V^{-1} &= \alpha S^t\theta U^t \\ &= \hat{T}_I^t\end{aligned} \tag{71b}$$

This gives

$$\begin{aligned}\hat{T}_V^{-1}L\hat{T}_I &= \alpha S^t\theta U^tLU\theta S\alpha \\ &= \alpha^2\Lambda^2\end{aligned} \tag{72a}$$

$$\begin{aligned}\hat{T}_I^{-1}C\hat{T}_V &= \alpha^{-1}S^t\theta^{-1}U^tCU\theta^{-1}S\alpha^{-1} \\ &= \alpha^{-2}.\end{aligned} \tag{72b}$$

Thus $\hat{T}_V^{-1}\hat{Z}\hat{Y}\hat{T}_V = \hat{T}_I^{-1}\hat{Y}\hat{Z}\hat{T}_I = -\omega^2\Lambda^2$ and

$$\hat{\gamma} = j\omega\Lambda. \tag{73}$$

Therefore

$$\begin{aligned}\hat{T}_V^{-1}\hat{Z}\hat{T}_I &= j\omega\alpha^2\Lambda^2 \\ &= \hat{d}\hat{\gamma} \end{aligned}\tag{74a}$$

$$\begin{aligned}\hat{T}_I^{-1}\hat{Y}\hat{T}_V &= j\omega\alpha^{-2} \\ &= \hat{d}^{-1}\hat{\gamma}\end{aligned}\tag{74b}$$

and $\hat{z}$ and $\hat{y}$ are diagonal. Hence we obtain

$$\hat{d} = \alpha^2\Lambda \tag{75a}$$

$$\hat{T}_V^t\hat{T}_I = \mathbf{1}_n. \tag{75b}$$

Thus

$$\hat{D}_V = \alpha\Lambda^{1/2} \tag{76a}$$

$$\hat{D}_I = \hat{D}_V^{-1} \tag{76b}$$

and the transformations in (50) become

$$\hat{M}_V = U\theta^{-1}S\Lambda^{1/2} \tag{77a}$$

$$\hat{M}_I = U\theta S\Lambda^{-1/2}. \tag{77b}$$

Again the power flow relation in (55) holds and $\hat{M}_V^t\hat{M}_I^* = \mathbf{1}_n$. One can also verify that the identities in (46) hold. The characteristic impedance matrix becomes

$$\hat{Z}_C = U\theta^{-1}S\Lambda S^t\theta^{-1}U^t. \tag{78}$$

This method was obtained previously in [13], [14], [27] and is a simple extension of well-known results [52]–[54]. It is equivalent to the method of [49]. The coupled microstrip is a common example of this case.

C. Lossy Conductors in Lossy, Homogeneous Media

Consider the case where we permit imperfect conductors, $R \neq 0$, but assume a homogeneous medium. The matrix product $\hat{Y}\hat{Z}$ becomes, using the identities for a homogeneous medium given in (3)

$$\begin{aligned}\hat{Y}\hat{Z} &= GR + j\omega CR + (j\omega\mu\sigma - \omega^2\mu\varepsilon)\mathbf{1}_n \\ &= \left(\frac{\sigma}{\varepsilon} + j\omega\right)CR + (j\omega\mu\sigma - \omega^2\mu\varepsilon)\mathbf{1}_n\end{aligned} \tag{79}$$

and we have neglected the internal inductance of the conductors. Hence we need only diagonalize CR as

$$\hat{T}^{-1}CR\hat{T} = \Lambda^2 \tag{80}$$

where Λ^2 is a diagonal matrix with Λ_i^2 on the main diagonal. The eigenvalues $\hat{Y}\hat{Z}$ become

$$\hat{\gamma}_i^2 = \left(\frac{\sigma}{\varepsilon} + j\omega\right)\Lambda_i^2 + (j\omega\mu\sigma - \omega^2\mu\varepsilon). \tag{81}$$

But this is virtually identical to the previous problem where we interchange the roles of L and R. However, R is frequency dependent as is $\hat{T}$ (which is real). Although this poses no computational problems for frequency-domain calculations it does create problems in the direct time-domain solution. If either all $(n+1)$ conductors are identical or the n conductors are identical and the reference conductor losses are ignored, then (80) requires that we only diagonalize C so that the transformation is frequency independent. Lossy, coupled striplines can be handled with this development.

D. Cyclic-Symmetric Structures

The MTL structures considered in the previous sections are such that the matrix products $\hat{Y}\hat{Z}$ and $\hat{Z}\hat{Y}$ can always be diagonalized with a numerically efficient and stable similarity transformation. The transformations are real and, with the exception of the case in section (c), frequency independent and can be directly applied to the time-domain solution. Not all structures can be diagonalized in this fashion. One can attempt to diagonalize $\hat{Z}\hat{Y}$ or $\hat{Y}\hat{Z}$ with a digital computer subroutine that determines the eigenvalues and eigenvectors of a general, complex matrix but we are not assured that a linearly independent set can be found that diagonalizes the matrix. Furthermore, the transformation matrices will be frequency dependent. This section discusses MTL's which have certain structural symmetry so that a numerically stable (and trivial) transformation can be always be found which diagonalizes $\hat{Z}\hat{Y}$ and $\hat{Y}\hat{Z}$. Furthermore this transformation is frequency independent regardless of whether the line is lossy and/or the medium is inhomogeneous, i.e., the general case.

Consider structures composed of n identical conductors and a reference conductor wherein the n conductors have structural symmetry with respect to the reference conductor so that the per-unit-length impedance and admittance matrices have the following structural symmetry [13], [29]:

$$\hat{Z} = \begin{bmatrix} \hat{Z}_1 & \hat{Z}_2 & \hat{Z}_3 & \cdots & \hat{Z}_3 & \hat{Z}_2 \\ \hat{Z}_2 & \hat{Z}_1 & \hat{Z}_2 & \hat{Z}_3 & \ddots & \hat{Z}_3 \\ \hat{Z}_3 & \hat{Z}_2 & \hat{Z}_1 & \hat{Z}_2 & \ddots & \vdots \\ \vdots & \hat{Z}_3 & \hat{Z}_2 & \ddots & \ddots & \hat{Z}_3 \\ \hat{Z}_3 & \ddots & \ddots & \ddots & \hat{Z}_1 & \hat{Z}_2 \\ \hat{Z}_2 & \hat{Z}_3 & \cdots & \hat{Z}_3 & \hat{Z}_2 & \hat{Z}_1 \end{bmatrix} \tag{82}$$

and $\hat{Y}$ has a similar form. A general cyclic-symmetric matrix $\hat{M}$ has the entries given by

$$[\hat{M}]_{ij} = \hat{M}_{|i-j|+1} \tag{83a}$$

where

$$\hat{M}_{j\pm n} = \hat{M}_j \tag{83b}$$

$$\hat{M}_{n+2-j} = \hat{M}_j \tag{83c}$$

and indices greater than n or less than 1 are defined by the convention: $n + j = j$ and $n + i = i$. Because of this special structure of the per-unit-length matrices, they are normal matrices, $\hat{Z}\hat{Z}^{t^*} = \hat{Z}^{t^*}\hat{Z}$, and we are guaranteed that each can be diagonalized as [53]

$$\hat{T}^{-1}\hat{Z}\hat{T} = \hat{\gamma}_Z^2 \tag{84a}$$

$$\hat{T}^{-1}\hat{Y}\hat{T} = \hat{\gamma}_Y^2 \tag{84b}$$

where the $n \times n$ matrices $\hat{\gamma}_Z^2$ and $\hat{\gamma}_Y^2$ are diagonal whose diagonal entries are given by

$$[\hat{\gamma}_Z^2]_{ii} = \left[\sum_{p=1}^{n}[\hat{\boldsymbol{Z}}]_{1p}\angle^{\{(2\pi/n)(p-1)(i-1)\}}\right] \tag{85a}$$

$$[\hat{\gamma}_Y^2]_{ii} = \left[\sum_{p=1}^{n}[\hat{\boldsymbol{Y}}]_{1p}\angle^{\{(2\pi/n)(p-1)(i-1)\}}\right]. \tag{85b}$$

Hence the first-order equations in (5) are uncoupled. The transformation is trivial to obtain

$$[\hat{\boldsymbol{T}}]_{ij} = \frac{1}{\sqrt{n}}\angle^{\{(2\pi/n)(i-1)(j-1)\}} \tag{86a}$$

and $\hat{\boldsymbol{T}}$ is unitary and symmetric:

$$\begin{aligned}\hat{\boldsymbol{T}}^{-1} &= \hat{\boldsymbol{T}}^{t^*}\\ &= \hat{\boldsymbol{T}}^{*}.\end{aligned} \tag{86b}$$

The square roots of the eigenvalues of $\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}$ and $\hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}$ (the propagation constants) are then $\hat{\gamma} = \hat{\gamma}_Z\hat{\gamma}_Y$. Observe that

$$\begin{aligned}\hat{\boldsymbol{T}}^t\hat{\boldsymbol{T}} &= \hat{\boldsymbol{T}}^2\\ &= \hat{\boldsymbol{D}}\end{aligned} \tag{87a}$$

where $\hat{\boldsymbol{D}}$ has a particularly nice form

$$\hat{\boldsymbol{D}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & \cdots & 1 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 1 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\end{bmatrix}. \tag{87b}$$

Hence we may define, according to (20)

$$\begin{aligned}\hat{\boldsymbol{T}}_V &= \hat{\boldsymbol{T}}(\hat{\boldsymbol{D}}^{-1})^t\\ &= \hat{\boldsymbol{T}}^{-1}\end{aligned} \tag{88a}$$

$$\hat{\boldsymbol{T}}_I = \hat{\boldsymbol{T}} \tag{88b}$$

and $\hat{\boldsymbol{T}}_V^t\hat{\boldsymbol{T}}_I = \boldsymbol{1}_n$. According to (17) $\hat{\boldsymbol{D}}$ commutes with $\hat{\gamma}^2$ so that we have the identity $\hat{\gamma}^2\hat{\boldsymbol{D}} = \hat{\boldsymbol{D}}\hat{\gamma}^2$. In addition, because of (87a) $\hat{\boldsymbol{D}}$ is symmetric, $\hat{\boldsymbol{D}}^t = \hat{\boldsymbol{D}}$, and $\hat{\boldsymbol{D}}^{-1} = \hat{\boldsymbol{D}}$. Therefore

$$\begin{aligned}\hat{\boldsymbol{D}}_V &= \hat{\boldsymbol{d}}^{1/2}\\ &= (\hat{\boldsymbol{D}}\hat{\gamma}_Z\hat{\gamma}_Y^{-1})^{1/2}\\ &= \hat{\boldsymbol{D}}^{1/2}\hat{\gamma}_Z^{1/2}\hat{\gamma}_Y^{-1/2}\end{aligned} \tag{89a}$$

$$\begin{aligned}\hat{\boldsymbol{D}}_I &= \hat{\boldsymbol{d}}^{-1/2}\\ &= (\hat{\boldsymbol{D}}\hat{\gamma}_Z\hat{\gamma}_Y^{-1})^{-1/2}\\ &= \hat{\boldsymbol{D}}^{-1/2}\hat{\gamma}_Z^{-1/2}\hat{\gamma}_Y^{1/2}\end{aligned} \tag{89b}$$

and the transformations in (48) become

$$\hat{\boldsymbol{M}}_V = \hat{\boldsymbol{T}}\hat{\boldsymbol{D}}^{-1/2}\hat{\gamma}_Z^{1/2}\hat{\gamma}_Y^{-1/2} \tag{90a}$$

$$\hat{\boldsymbol{M}}_I = \hat{\boldsymbol{T}}\hat{\boldsymbol{D}}^{-1/2}\hat{\gamma}_Z^{-1/2}\hat{\gamma}_Y^{1/2}. \tag{90b}$$

The transformation is not unitary, so that the power flow relation in (55) does not hold. However, $\hat{\boldsymbol{T}}_V^t\hat{\boldsymbol{T}}_I^* = \hat{\boldsymbol{D}}^{-1} = \hat{\boldsymbol{D}}$. Because of the form of $\hat{\boldsymbol{D}}$ given in (87b) the total power flow on the line is a simple combination of the power in the modes. One can also verify that the identities in (46) hold. The characteristic impedance matrix becomes

$$\hat{\boldsymbol{Z}}_C = \hat{\boldsymbol{T}}\hat{\gamma}_Z\hat{\gamma}_Y^{-1}\hat{\boldsymbol{T}}^*. \tag{91}$$

As an illustration of these results consider a four-conductor line ($n = 3$) with cyclic symmetric structure such that

$$\hat{\boldsymbol{Z}} = \begin{bmatrix}\hat{Z}_s & \hat{Z}_m & \hat{Z}_m\\ \hat{Z}_m & \hat{Z}_s & \hat{Z}_m\\ \hat{Z}_m & \hat{Z}_m & \hat{Z}_s\end{bmatrix} \tag{92}$$

and $\hat{\boldsymbol{Y}}$ has a similar form. The matrix $\hat{\boldsymbol{T}}$ which decouples the equations becomes

$$\hat{\boldsymbol{T}} = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & e^{j2\pi/3} & e^{j4\pi/3}\\ 1 & e^{j4\pi/3} & e^{j2\pi/3}\end{bmatrix} \tag{93a}$$

and $\hat{\boldsymbol{D}}$ has the form

$$\hat{\boldsymbol{D}} = \hat{\boldsymbol{T}}^t\hat{\boldsymbol{T}} = \hat{\boldsymbol{T}}^2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \tag{93b}$$

whose square root is

$$\hat{\boldsymbol{D}}^{1/2} = \begin{bmatrix}1 & 0 & 0\\ 0 & \dfrac{1}{\sqrt{2j}} & \dfrac{j}{\sqrt{2j}}\\ 0 & \dfrac{j}{\sqrt{2j}} & \dfrac{1}{\sqrt{2j}}\end{bmatrix}. \tag{93c}$$

Hence $\hat{\boldsymbol{T}}_V = \hat{\boldsymbol{T}}^{-1} = \hat{\boldsymbol{T}}^*$ and $\hat{\boldsymbol{T}}_I = \hat{\boldsymbol{T}}$. The propagation constants become

$$\hat{\gamma}_1 = \sqrt{(\hat{Z}_s + 2\hat{Z}_m)(\hat{Y}_s + 2\hat{Y}_m)} \tag{94a}$$

$$\hat{\gamma}_2 = \hat{\gamma}_3 = \sqrt{(\hat{Z}_s - \hat{Z}_m)(\hat{Y}_s - \hat{Y}_m)} \tag{94b}$$

and two of the propagation constants are equal. Define the characteristic impedances of the modes as

$$\hat{Z}_C^+ = \sqrt{(\hat{Z}_s + 2\hat{Z}_m)/(\hat{Y}_s + 2\hat{Y}_m)} \tag{95a}$$

$$\hat{Z}_C^- = \sqrt{(\hat{Z}_s - \hat{Z}_m)/(\hat{Y}_s - \hat{Y}_m)} \tag{95b}$$

yielding

$$\hat{\boldsymbol{d}} = \hat{\boldsymbol{D}}\hat{\gamma}_Z\hat{\gamma}_Y^{-1} = \begin{bmatrix}\hat{Z}_C^+ & 0 & 0\\ 0 & 0 & \hat{Z}_C^-\\ 0 & \hat{Z}_C^- & 0\end{bmatrix}. \tag{96}$$

The characteristic impedance matrix becomes

$$\begin{aligned}\hat{\boldsymbol{Z}}_C &= \hat{\boldsymbol{Z}}\hat{\boldsymbol{T}}_I\hat{\gamma}^{-1}\hat{\boldsymbol{T}}_I^{-1}\\ &= \hat{\boldsymbol{T}}_V\hat{\boldsymbol{d}}\hat{\boldsymbol{T}}_I^{-1}\\ &= \hat{\boldsymbol{T}}\hat{\gamma}_Z\hat{\gamma}_Y^{-1}\hat{\boldsymbol{T}}^*\\ &= \tfrac{1}{3}\begin{bmatrix}(\hat{Z}_C^+ + 2\hat{Z}_C^-) & (\hat{Z}_C^+ - \hat{Z}_C^-) & (\hat{Z}_C^+ - \hat{Z}_C^-)\\ (\hat{Z}_C^+ - \hat{Z}_C^-) & (\hat{Z}_C^+ + 2\hat{Z}_C^-) & (\hat{Z}_C^+ - \hat{Z}_C^-)\\ (\hat{Z}_C^+ - \hat{Z}_C^-) & (\hat{Z}_C^+ - \hat{Z}_C^-) & (\hat{Z}_C^+ + 2\hat{Z}_C^-)\end{bmatrix}.\end{aligned} \tag{97}$$

We obtain

$$\hat{\boldsymbol{D}}_V = \begin{bmatrix} \hat{Z}_C^+ & 0 & 0 \\ 0 & 0 & \hat{Z}_C^- \\ 0 & \hat{Z}_C^- & 0 \end{bmatrix}^{1/2} = \begin{bmatrix} \sqrt{\hat{Z}_C^+} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2j}}\sqrt{\hat{Z}_C^-} & \frac{j}{\sqrt{2j}}\sqrt{\hat{Z}_C^-} \\ 0 & \frac{j}{\sqrt{2j}}\sqrt{\hat{Z}_C^-} & \frac{1}{\sqrt{2j}}\sqrt{\hat{Z}_C^-} \end{bmatrix} \quad (98a)$$

$$\hat{\boldsymbol{D}}_I = \hat{\boldsymbol{D}}_V^{-1} = \begin{bmatrix} \sqrt{\hat{Y}_C^+} & 0 & 0 \\ 0 & \frac{j}{\sqrt{2j}}\sqrt{\hat{Y}_C^-} & \frac{1}{\sqrt{2j}}\sqrt{\hat{Y}_C^-} \\ 0 & \frac{1}{\sqrt{2j}}\sqrt{\hat{Y}_C^-} & \frac{j}{\sqrt{2j}}\sqrt{\hat{Y}_C^-} \end{bmatrix} \quad (98b)$$

where $\hat{Z}_C^\pm = 1/\hat{Y}_C^\pm$. The transformations become $\hat{\boldsymbol{M}}_I = \hat{\boldsymbol{T}}\hat{\boldsymbol{D}}_I$, $\hat{\boldsymbol{M}}_V = \hat{\boldsymbol{T}} * \hat{\boldsymbol{D}}_V$.

There are a number of cases where a MTL can be approximated as a cyclic-symmetric structure. A common case is a three-phase, high-voltage power transmission line consisting of three wires above earth. Assuming a balanced line wherein the conductors are transposed at regular intervals, the per-unit-length matrices, $\hat{\boldsymbol{Z}}$ and $\hat{\boldsymbol{Y}}$, take on a cyclic-symmetric structure. The transformation is referred to in the power transmission literature as the method of symmetrical components [5]. In the case of unbalanced lines where, for example, one phase may be shorted to ground, this transformation does not apply. Other approximations of MTL's as cyclic-symmetric structures are useful. Cable harnesses carrying tightly-packed, insulated wires have been assumed to be cyclic symmetric structures on the notion that all wires occupy at some point along the line all possible positions [29]. This leads to a cyclic-symmetric structure of the $n \times n$ per-unit-length impedance and admittance matrices that is similar to the special case of transposed power distribution lines in that all off-diagonal terms are equal and, if we assume the conductors are identical, the main diagonal terms are equal.

Other common cases are the cyclic-symmetric, three-conductor lines consisting of two identical conductors above a ground plane such as the coupled microstrip. The per-unit-length impedance and admittance matrices become

$$\hat{\boldsymbol{Z}} = \begin{bmatrix} \hat{Z}_s & \hat{Z}_m \\ \hat{Z}_m & \hat{Z}_s \end{bmatrix} \quad (99a)$$

$$\hat{\boldsymbol{Y}} = \begin{bmatrix} \hat{Y}_s & \hat{Y}_m \\ \hat{Y}_m & \hat{Y}_s \end{bmatrix}. \quad (99b)$$

The transformation matrix and propagation constants simplify to

$$\hat{\boldsymbol{T}}_I = \hat{\boldsymbol{T}}_V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (100a)$$

$$\hat{\gamma}_1 = \sqrt{(\hat{Z}_s + \hat{Z}_m)(\hat{Y}_s + \hat{Y}_m)} \quad (100b)$$

$$\hat{\gamma}_2 = \sqrt{(\hat{Z}_s - \hat{Z}_m)(\hat{Y}_s - \hat{Y}_m)}. \quad (100c)$$

This transformation is very common in the microwave literature and is referred to as the even-odd mode transformation. Defining the even-odd mode characteristic impedances as

$$\hat{Z}_C^\pm = \sqrt{\frac{\hat{Z}_s \pm \hat{Z}_m}{\hat{Y}_m \pm \hat{Y}_m}} \quad (101)$$

we obtain

$$\hat{\boldsymbol{d}} = \begin{bmatrix} \hat{Z}_C^+ & 0 \\ 0 & \hat{Z}_C^- \end{bmatrix} \quad (102)$$

and $\hat{\boldsymbol{D}} = \boldsymbol{1}_2$. The characteristic impedance matrix becomes

$$\hat{\boldsymbol{Z}}_C = \frac{1}{2}\begin{bmatrix} (\hat{Z}_C^+ + \hat{Z}_C^-) & (\hat{Z}_C^+ - \hat{Z}_C^-) \\ (\hat{Z}_C^+ - \hat{Z}_C^-) & (\hat{Z}_C^+ + \hat{Z}_C^-) \end{bmatrix}. \quad (103)$$

Also

$$\hat{\boldsymbol{M}}_V = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{\hat{Z}_C^+} & \sqrt{\hat{Z}_C^-} \\ \sqrt{\hat{Z}_C^+} & -\sqrt{\hat{Z}_C^-} \end{bmatrix} \quad (104a)$$

$$\hat{\boldsymbol{M}}_I = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{\hat{Y}_C^+} & \sqrt{\hat{Y}_C^-} \\ \sqrt{\hat{Y}_C^+} & -\sqrt{\hat{Y}_C^-} \end{bmatrix}. \quad (104b)$$

The power relation in (55) as well as the identities in (46) are satisfied.

E. The General Case

The results of this paper assume that a similarity transformation can be found which diagonalizes the products of $\hat{\boldsymbol{Z}}$ and $\hat{\boldsymbol{Y}}$. This is accomplished for the above cases but for others one may or may not find n linearly independent eigenvectors. Although other cases may not be diagonalizable they can be reduced via a similarity transformation to more convenient forms. For example, all matrices can be reduced to the Jordan canonical form with the eigenvalues on the main diagonal, one's in selected positions on the upper diagonal, and zero's elsewhere [52]–[54]. Although not completely decoupled, the solution of the second-order MTL equations is considerably simplified [27]. Also one can always find a unitary transformation that reduces any general matrix to upper triangular form [53]. This reduction, while not completely decoupling the MTL equations can result in a simplification of the general solution.

V. Reflection Coefficient Matrices

It is common to define a voltage reflection coefficient matrix, $\hat{\Gamma}_V(z)$, by analogy to the two-conductor line. The expressions for the phasor voltages and currents in (13) can be written in the form of forward-traveling waves $\hat{\boldsymbol{V}}^+(z)$ and $\hat{\boldsymbol{I}}^+(z)$, and backward-traveling waves, $\hat{\boldsymbol{V}}^-(z)$ and $\hat{\boldsymbol{I}}^-(z)$, as

$$\hat{\boldsymbol{V}}(z) = \hat{\boldsymbol{V}}^+(z) + \hat{\boldsymbol{V}}^-(z) \quad (105a)$$

$$\hat{\boldsymbol{I}}(z) = \hat{\boldsymbol{I}}^+(z) - \hat{\boldsymbol{I}}^-(z) \quad (105b)$$

where $\hat{\boldsymbol{V}}^\pm(z) = \hat{\boldsymbol{T}}_V e^{\mp\hat{\gamma}z}\hat{\boldsymbol{V}}_m^\pm$ and $\hat{\boldsymbol{I}}^\pm(z) = \hat{\boldsymbol{T}}_I e^{\pm\hat{\gamma}z}\hat{\boldsymbol{I}}_m^\pm$. As was shown previously, the forward and backward traveling voltage and current waves are related by the characteristic impedance matrix as $\hat{\boldsymbol{V}}^\pm(z) = \hat{\boldsymbol{Z}}_C\hat{\boldsymbol{I}}^\pm(z)$. Define the voltage reflection coefficient matrix, $\hat{\Gamma}_V(z)$, in a logical manner relating the reflected or backward-traveling voltage waves to the incident or forward-traveling voltage waves at any point on the line yields

$$\hat{\boldsymbol{V}}^{-}(z) = \hat{\Gamma}_V(z)\hat{\boldsymbol{V}}^{+}(z). \tag{106}$$

Also

$$\hat{\boldsymbol{I}}^{-}(z) = \hat{\boldsymbol{Y}}_C\hat{\Gamma}_V(z)\hat{\boldsymbol{Z}}_C\hat{\boldsymbol{I}}^{+}(z). \tag{107}$$

Hence, the current reflection coefficient matrix becomes

$$\hat{\Gamma}_I(z) = -\hat{\boldsymbol{Y}}_C\hat{\Gamma}_V(z)\hat{\boldsymbol{Z}}_C \tag{108}$$

since the total backward-traveling current is $-\hat{\boldsymbol{I}}^{-}(z)$. The reflection coefficient matrices are, in general, not symmetric. Although (108) reduces to the scalar case for two-conductor lines, it is important to distinguish between the two reflection coefficient matrices in the MTL case since the products in (108) do not commute. Substituting (106) into (105) gives

$$\begin{aligned}\hat{\boldsymbol{V}}(z) &= [\mathbf{1}_n + \hat{\Gamma}_V(z)]\hat{\boldsymbol{V}}^{+}(z)\\ &= [\mathbf{1}_n - \hat{\boldsymbol{Z}}_C\hat{\Gamma}_I(z)\hat{\boldsymbol{Y}}_C]\hat{\boldsymbol{V}}^{+}(z) \qquad (109a)\\ \hat{\boldsymbol{I}}(z) &= \hat{\boldsymbol{Y}}_C[\mathbf{1}_n - \hat{\Gamma}_V(z)]\hat{\boldsymbol{V}}^{+}(z)\\ &= [\mathbf{1}_n + \hat{\boldsymbol{Z}}_C\hat{\Gamma}_I(z)\hat{\boldsymbol{Y}}_C]\hat{\boldsymbol{V}}^{+}(z). \qquad (109b)\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\hat{\boldsymbol{V}}(z) &= \hat{\boldsymbol{Z}}_C[\mathbf{1}_n + \hat{\boldsymbol{Y}}_C\hat{\Gamma}_V(z)\hat{\boldsymbol{Z}}_C]\hat{\boldsymbol{I}}^{+}(z)\\ &= \hat{\boldsymbol{Z}}_C[\mathbf{1}_n - \hat{\Gamma}_I(z)]\hat{\boldsymbol{I}}^{+}(z) \qquad (110a)\\ \hat{\boldsymbol{I}}(z) &= [\mathbf{1}_n - \hat{\boldsymbol{Y}}_C\hat{\Gamma}_V(z)\hat{\boldsymbol{Z}}_C]\hat{\boldsymbol{I}}^{+}(z)\\ &= [\mathbf{1}_n + \hat{\Gamma}_I(z)]\hat{\boldsymbol{I}}^{+}(z). \qquad (110b)\end{aligned}$$

The input impedance matrix at any point on the line relates the total voltages and total currents at that point as

$$\hat{\boldsymbol{V}}(z) = \boldsymbol{Z}_{\text{in}}(z)\hat{\boldsymbol{I}}(z). \tag{111}$$

Substituting (109) and (110) into (111) yields

$$\begin{aligned}\hat{\boldsymbol{Z}}_{\text{in}} &= [\mathbf{1}_n + \hat{\Gamma}_V(z)][\mathbf{1}_n - \hat{\Gamma}_V(z)]^{-1}\hat{\boldsymbol{Z}}_C\\ &= \hat{\boldsymbol{Z}}_C[\mathbf{1}_n + \hat{\boldsymbol{Y}}_C\hat{\Gamma}_V(z)\hat{\boldsymbol{Z}}_C][\mathbf{1}_n - \hat{\boldsymbol{Y}}_C\hat{\Gamma}_V(z)\hat{\boldsymbol{Z}}_C]^{-1}\\ &= \hat{\boldsymbol{Z}}_C[\mathbf{1}_n - \hat{\Gamma}_I(z)][\mathbf{1}_n + \hat{\Gamma}_I(z)]^{-1}. \qquad (112)\end{aligned}$$

Similarly, the voltage reflection coefficient matrix can be written in terms of the input impedance matrix at a point on the line from (112) as

$$\begin{aligned}\hat{\Gamma}_V(z) &= \hat{\boldsymbol{Z}}_C[\hat{\boldsymbol{Z}}_{\text{in}}(z) + \hat{\boldsymbol{Z}}_C]^{-1}[\hat{\boldsymbol{Z}}_{\text{in}}(z) - \hat{\boldsymbol{Z}}_C]\hat{\boldsymbol{Z}}_C^{-1}\\ &= [\hat{\boldsymbol{Z}}_{\text{in}}(z) - \hat{\boldsymbol{Z}}_C][\hat{\boldsymbol{Z}}_{\text{in}}(z) + \hat{\boldsymbol{Z}}_C]^{-1}. \qquad (113)\end{aligned}$$

From the relation for the current reflection coefficient given by (108) and observing (113) we see that the current reflection coefficient matrix is

$$\begin{aligned}\hat{\Gamma}_I(z) &= -\hat{\boldsymbol{Z}}_C^{-1}\hat{\Gamma}_V(z)\hat{\boldsymbol{Z}}_C\\ &= -[\hat{\boldsymbol{Z}}_{\text{in}}(z) + \hat{\boldsymbol{Z}}_C]^{-1}[\hat{\boldsymbol{Z}}_{\text{in}}(z) - \hat{\boldsymbol{Z}}_C]\\ &= -\hat{\boldsymbol{Z}}_C^{-1}[\hat{\boldsymbol{Z}}_{\text{in}}(z) - \hat{\boldsymbol{Z}}_C][\hat{\boldsymbol{Z}}_{\text{in}}(z) + \hat{\boldsymbol{Z}}_C]^{-1}\hat{\boldsymbol{Z}}_C. \qquad (114)\end{aligned}$$

Although the above formulas reduce to the corresponding scalar results for a two-conductor line, the MTL results are considerably more complicated. In addition, one must distinguish between whether the voltage reflection coefficient matrix or the current reflection matrix is being used since the relation between the two is not simple. This has also led to misunderstanding and confusion.

VI. Summary and Conclusion

This paper has attempted to clarify some of the problems associated with the decoupling of the phasor MTL equations. Diagonalizing the matrix products $\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}$ and/or $\hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}$ with similarity transformations as $\hat{\boldsymbol{T}}_V^{-1}\hat{\boldsymbol{Z}}\hat{\boldsymbol{Y}}\hat{\boldsymbol{T}}_V = \hat{\gamma}^2$ or $\hat{\boldsymbol{T}}_I^{-1}\hat{\boldsymbol{Y}}\hat{\boldsymbol{Z}}\hat{\boldsymbol{T}}_I = \hat{\gamma}^2$ may or may not be possible. However, when it is possible, the solution to the MTL equations is straightforward. We have shown a number of common cases where this diagonalization can be accomplished. For those cases where the diagonalization is assured, it can be accomplished with a numerically stable and frequency-independent transformation. Additional topics analogous to the common two-conductor line for the MTL case such as the voltage and current reflection coefficient matrices and the input impedance matrix were also discussed. Although these quantities reduce to the familiar results for a two-conductor line, in the case of a MTL they are not as simple and care must be maintained to observe the proper order of matrix multiplication.

We also examined the common redefinitions of the transformations which have led to confusion in the literature. The keys to obtaining correct quantities such as the characteristic impedance matrix with the alternative transformations are to arrange the eigenvalues and eigenvectors as in (14) and (15) and normalize the eigenvectors such that $\hat{\boldsymbol{T}}_V^t\hat{\boldsymbol{T}}_I = \mathbf{1}_n$ while retaining the ability to decouple the second-order equations. This leads to a consistent redefinition of the transformation matrices $\hat{\boldsymbol{M}}_{V,I}$. The topic is not difficult but can be made so if one defines quantities not directly necessary to achieve the decoupling of the MTL equations. This has led to considerable confusion and the results of this paper are intended to point out where the trouble arises.

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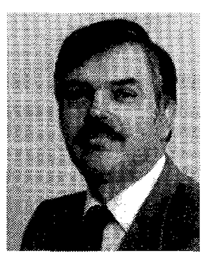

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